Bi-Intuitionism as dialogue chirality

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Correctness by Construction Workshop, March 25, 2014

Plan of the talk

1. Rauszer's Bi-Intuitionism
2. No categorical semantics for Rauszer's Bi-Intuitionism
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C. Rauszer's Bi-Intuitionism

Heyting algebra a bounded lattice $\mathscr{A} = (A, \vee, \wedge, 0, 1)$ with *Heyting implication* $(\rightarrow)$, defined as the right adjoint to meet.

$$\frac{c \wedge b \leq a}{c \leq b \rightarrow a}$$

co-Heyting algebra is a lattice $\mathscr{C}$ such that $\mathscr{C}^{op}$ is a Heyting algebra: $\mathscr{C} = (C, \vee, \wedge, 1, 0)$ with *subtraction* $(\prec)$ defined as the left adjoint of join.

$$\frac{a \leq b \vee c}{a \prec b \leq c}$$

Bi-Heyting algebra: a lattice with the structure of Heyting and of co-Heyting algebra.

C. Rauszer's Bi-Intuitionism

Bi-intuitionistic language

$A, B := a \mid \top \mid \bot \mid A \wedge B \mid A \rightarrow B \mid A \vee B \mid A \prec B$

Read $A \prec B$ as "A excludes B".

Two negations:

- strong intuitionistic negation $\sim A =_{df} A \rightarrow \bot$
- weak co-intuitionistic negation $\frown A =_{df} \top \prec A$

  to be distinguished from classical negation $\neg A$

F. W. Lawvere. Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes, *Category Theory (Como 1990)*, LNM 1488, 1991
[Reyes and Zolfaghari 1996] [Stell and Worboys 1997][Pagliani 1998]

Kripke models [Rauszer 1977]

$(W, \leq, \Vdash)$, with $(W, \leq)$ a preorder:

- $w \Vdash A \rightarrow B$ iff $\forall w' \geq w. w' \Vdash A$ implies $w' \Vdash B$;
- $w \Vdash A \prec B$ iff $\exists w' \leq w. w' \Vdash A$ and not $w' \Vdash B$.

C. Rauszer's Bi-Intuitionism

How to formalize Bi-intuitionism in a Gentzen system?

$$\to\text{-R}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}\ (*) \qquad\qquad \to\text{-L}\ \frac{\Gamma_1 \Rightarrow \Delta_1 A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, A \to B, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\smallsetminus\text{-R}\ \frac{\Gamma_1 \Rightarrow \Delta_1, C \qquad D, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, C \smallsetminus D, \Delta_2} \qquad\qquad \smallsetminus\text{-L}\ \frac{C \vdash D, \Delta}{\Gamma, C \smallsetminus D \Rightarrow \Delta}\ (**)$$

- The formalization is not trivial (see [Crolard 2001]).
- Cut elimination is problematic.

No categorical model for Rauszer's logic

Let $\mathscr{C}$ be a CCC and let $\bot$ be initial in $\mathscr{C}$.

Theorem (Joyal's Theorem)

For any object A in $\mathscr{C}$, if $\mathscr{C}(A, \bot)$ is nonempty, then A is initial.

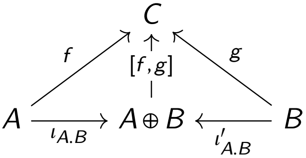
Proof.

$\bot \times A$ is initial, as $\mathscr{C}((\bot \times A), B) \approx \mathscr{C}(\bot, B^A)$. Given $f : A \to \bot$, show that $A \approx \bot \times A$, using the fact that $\langle f, id_A \rangle \circ \pi'_{\bot,A} = id_{\bot,A}$, since $\bot \times A$ is initial. □

No categorical model for Rauszer's logic

Definition (Coproduct)

The *coproduct* of A and B is an object $A \oplus B$ together with arrows $\iota_{A,B}$ and $\iota'_{A,B}$ such that for every C and every pair of arrows $f : A \to C$ and $g : B \to C$ there is a unique $[f, g] : A \oplus B \to C$ making the following diagram commute:



No categorical model for Rauszer's logic

Definition (Coexponent)

The *coexponent* of A and B is an object B_A together with an arrow $\backepsilon_{A,B} : B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ making the following diagram commute:

$$
\begin{array}{ccccc}
B & \xrightarrow{f} & C \oplus A & \qquad & C \\
& \searrow^{\backepsilon_{A,B}} & \uparrow{\scriptstyle f_* \oplus id_A} & & \uparrow{\scriptstyle f_*} \\
& & B_A \oplus A & & B_A
\end{array}
$$

No categorical model for Rauszer's logic

Theorem (Crolard's Theorem)

If both $\mathscr{C}$ and $\mathscr{C}^{op}$ are CCCs, then $\mathscr{C}$ is a preorder.

Proof.

Let $A \oplus B$ be the *coproduct* and A_B the *coexponent* of A and B. Then $\mathscr{C}(A, B) \approx \mathscr{C}(A, \bot \oplus B) \approx \mathscr{C}(A_B, \bot)$. By Joyal's Theorem $\mathscr{C}(A_B, \bot)$ contains at most one arrow. □

No categorical model for Rauszer's logic

Theorem (Crolard's Lemma)

The coexponent B_A of two sets A and B is defined if and only if $A = \emptyset$ or $B = \emptyset$.

Proof.

In **Set** the *coproduct* is the disjoint union and the initial object is $\emptyset$.

(if) For any B, let $B_\perp =_{df} B$ with $\ni_{\perp,B} =_{df} \iota_{B,\perp}$. For any A, let $\perp_A =_{df} \perp$ with $\ni_{A,\perp} =_{df} \Box : \perp \to \perp \oplus A$.

(only if) If $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_* \oplus id_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute. □

No "perfect duality" between Intuitionism and co-Intuitionism

No modelling of **co-Int**. in **Set** because

- false ($\bot$) = the initial object and
- disjunction = coproduct.

"Perfect duality" in the linear case:

Multiplicative linear Int.: $\mathscr{A} = (A, 1, \otimes, \multimap)$ (with natural isomorphisms), *symmetric monoidal closed* (with $\multimap$ the right adjoint of $\otimes$).

Multiplicative linear co-Int.: $\mathscr{C} = (C, \bot, \wp, \smallsetminus)$ (with natural isomorphisms), *symmetric monoidal left-closed* (with $\smallsetminus$ the left adjoint of $\wp$).

Dialogue chirality

A dialogue chirality on the left is a pair of monoidal categories $(\mathscr{A}, \wedge, \mathrm{true})$ and $(\mathscr{B}, \vee, \mathrm{false})$ equipped with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

whose unit and counit are denoted as

$$\eta\colon\ id \to R \circ L,\ \epsilon\colon\ L \circ R \to id$$

together with a monoidal functor[1]

$$(-)^* : \mathscr{A} \to \mathscr{B}^{op(0,1)}$$

and a family of bijections

$$\chi_{m,a,b}\colon \langle m \wedge a | b \rangle \to \langle a | m^* \vee b \rangle$$

natural in m, a, b *(curryfication)*. Here the bracket $\langle a|b \rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle a|b \rangle = \mathscr{A}\,(a, R(b))$$

[1] In the context of 2-categories, the notation $\mathscr{B}^{op(0,1)}$ means that the *op* operation applies to 0-cells and 1-cells.

Dialogue chirality

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle (m \wedge n) \wedge a | b \rangle & & \xrightarrow[\chi_{m \wedge n}]{\hspace{12em}} & & \langle a | (m \wedge n)^* \vee b \rangle \\
\Big\downarrow \text{assoc.} & & = & & \Big\uparrow \text{assoc. monoid. of } (-)^* \\
\langle m \wedge (n \wedge a) | b \rangle & \xrightarrow{\chi_m} & \langle n \wedge a | m^* \vee b \rangle & \xrightarrow{\chi_n} & \langle a | n^* \vee (m^* \vee b) \rangle
\end{array}$$

commute for all objects a, m, n, all morphisms $f : m \to n$ of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$.

Dialogue chirality and Bi-intuitionism

Think of:

- $\mathscr{A}$ as a model of **Int conjunctive logic** on the language $\cap, \top$ ($\mathscr{A}$ may be Cartesian).
- $\mathscr{B}$ as a model of **co-Int disjunctive logic** on the language $\curlyvee, \bot$.
- The contravariant monoidal functor $()^* : \mathscr{A} \to \mathscr{B}^{op}$ models De Morgan duality.
- There is a dual contravariant functor $^*() : \mathscr{B} \to \mathscr{A}^{op}$.
- **What are the covariant functors** $L \dashv R$?
- **Main Idea:** introduce negations "$\sim$" in $\mathscr{A}$ and "$\frown$" in $\mathscr{B}$;
- let $L = \frown {}^*()$ and $R = \sim ()^*$.

Polarized Bi-Intuitionism $\mathbf{BI}_p$

Language of polarized bi-intuitionism $\mathbf{BI}_p$: sets of atoms $\{a_1, \ldots\}$ and $\{c_1, \ldots\}$;

$$A, B := a \mid \top \mid \mathbf{u} \mid A \cap B \mid \sim A \mid A \supset B \mid \sim C$$
$$C, D := c \mid \bot \mid \mathbf{j} \mid C \curlyvee D \mid \frown C \mid C \prec D \mid \frown A$$

Read $C \prec D$ as *"C excludes D"*.
Think of $\sim A =_{df} A \supset \mathbf{u}$, $\frown C =_{df} \mathbf{j} \prec C$; but in the chirality model $\sim A$ and $\frown C$ are primitive.

Polarized Bi-Intuitionism $\mathbf{BI}_p$

Informal interpretation

- **"Justification logic" of assertions and hypotheses;**
- **conclusive evidence** for assertions;
- **"scintilla of evidence"** for hypotheses.
- **Atoms:** $a_i = \vdash p_i$, $c_i = \mathscr{H} p_i$ (where p_i is a proposition).

- a_i is the type of evidence for assertions of p_i;
- c_j is the type of evidence for hypotheses on p_j;
- $A \supset B =$ the type of methods transforming evidence for A into evidence for B;
- $C \smallsetminus D =$ the type of hypothetical evidence that C is justified and D is refuted;
- $\mathbf{u} =$ an always unjustified assertion;
- $\mathbf{j} =$ an always justified hypothesis;
- $\sim X =$ denial of X; $\frown X =$ doubt about X, $X = A, C$.

Some problems

1. What is a *scintilla of evidence* and what is a *doubt about* an assertion or a hypothesis?
2. What does "*C excludes D*" mean?

Scintilla of evidence is legal terminology [Gordon and Walton 2009]. It evokes probabilistic methods, perhaps infinitely-valued logics (not discussed here).
An alternative: define **evidence for** and **evidence against** assertion and hypotheses. Obtain a "Dialectica-like" dialogue semantics [Bellin 2014].
A "non-logical axiom" (beyond the duality!): *If asserting p is justified, then it is justified making the hypothesis that p*

McKinsey-Tarski-Gödel's Translation

Modal S4 Translation

$$(\vdash p)^M = \Box p \qquad (\mathscr{H} p)^M = \Diamond p$$

$$(A \supset B)^M = \Box(A^M \to B^M) \qquad (C \smallsetminus D)^M = \Diamond(C^M \wedge \neg D^M)$$

$$(\top)^M = \mathbf{t} \qquad (\bot)^M = \mathbf{f}$$

$$(A \cap B)^M = A^M \wedge B^M \qquad (C \curlyvee D)^M = C^M \vee D^M$$

$$(\sim X)^M = \Box X^M \text{ for } X = A, C \qquad (\frown X)^M = \Diamond \neg X^M$$

Lemma

$A^M \equiv \Box A^M$, $C^M \equiv \Diamond C$.

Remarks

- $(\sim \mathbf{A})^M = \Box\neg\Box A^M = \Box\Diamond\neg\mathbf{A}^M$, $(\sim \mathbf{C})^M = \Box\neg\Diamond C^M = \Box\neg\mathbf{C}^M$. It distinguishes between negation and duality.
- $(C \smallsetminus D)^M = \Diamond(C^M \wedge \Box\neg D^M)$.

Proof-theoretic Meaning of Subtraction

Multiple-conclusion single-premise Natural Deduction

$$\frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \backsim D, \Delta}\ \backsim\text{-intro}$$

Computational meaning:
if $t\colon C$ and $x\colon D$, then $\texttt{make}-\texttt{coroutine}(t, x)\colon C \backsim D$.

$$\frac{H \vdash \Delta, C \backsim D \qquad C \vdash D, \Upsilon}{H \vdash \Delta, \Upsilon}\ \backsim\text{-elim}$$

Computational meaning:
if $u\colon C \backsim D$, $y\colon C$ and $t(y)\colon D$, then the term $\texttt{postpone}(y \mapsto f, u)$ is stored away.

Proof-theoretic Meaning of Subtraction

Normalization step for subtraction:

$$
\dfrac{\dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, C \smallsetminus D}\,{\smallsetminus}\text{-I} \qquad \begin{matrix} d_2 \\ C \vdash D, \curlyvee \end{matrix}}{H \vdash \Gamma, \Delta, \curlyvee}\,{\smallsetminus}\text{-E}
$$

reduces to

$$
\dfrac{\dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash D, \curlyvee \end{matrix}}{H \vdash \Gamma, D, \curlyvee}\,\mathit{subst} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, \curlyvee}\,\mathit{subst}
$$

See [Bellin and Menti 2014].

Sequent calculus for $\mathbf{BI}_p$

Two-zone sequents.

$$\Gamma; \Rightarrow A; \Delta \text{ or } \Gamma; C \Rightarrow; \Delta$$

$$\textbf{int: } \Gamma; \Rightarrow A; \ \textbf{co-int: } ; C \Rightarrow; \Delta$$

Write $\Gamma; \epsilon \Rightarrow \epsilon'; \Delta$, with exactly one of ϵ, ϵ' non-null.

Identity Rules

Logical axiom:
$A; \Rightarrow A;$

Logical axiom:
$; C \Rightarrow; C$

cut 1:

$$\frac{\Theta; \Rightarrow A; \Upsilon \qquad A, \Theta'; \epsilon \Rightarrow \epsilon'; \Upsilon'}{\Theta, \Theta'; \epsilon \Rightarrow \epsilon'; \Upsilon, \Upsilon'}$$

cut 2:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C \qquad \Theta'; C \Rightarrow \Upsilon'}{\Theta, \Theta'; \epsilon \Rightarrow \epsilon'; \Upsilon, \Upsilon'}$$

Sequent calculus for $\mathbf{BI}_p$

Duality Rules

$\sim$ right:

$$\frac{\Theta; C \Rightarrow ; \Upsilon}{\Theta; \Rightarrow \sim C; \Upsilon}$$

$\sim$ left:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C}{\sim C, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

$\frown$ right:

$$\frac{\Theta, A; \epsilon \Rightarrow \epsilon'; \Upsilon}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \frown A}$$

$\frown$ left:

$$\frac{\Theta; \Rightarrow A; \Upsilon}{\Theta; \frown A \Rightarrow ; \Upsilon}$$

$\mathbf{u}/\mathbf{j}$ left:

$$\mathbf{u}; \mathbf{j} \Rightarrow ;$$

$\mathbf{u}/\mathbf{j}$ right:

$$; \Rightarrow \mathbf{u}; \mathbf{j}$$

Sequent calculus for $\mathbf{BI}_p$

Asymmetric Non-Logical Axioms

$\vdash / \mathscr{H}$ left:	$\vdash / \mathscr{H}$ right:
$a_i ; \mathbf{j} \Rightarrow ; c_i$	$a_i ; \Rightarrow \mathbf{u} ; c_i$

where $a_i = \vdash p_i, \quad c_i = \mathscr{H} p_i.$

Sequent calculus for $\mathbf{BI}_p$

Structural rules

Contraction left:

$$\frac{A, A, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}{A, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

Contraction right:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C, C}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C}$$

Weakening left:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}{A, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

Weakening right:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C}$$

Sequent calculus for $\mathbf{BI}_p$

Conjunction and disjunction

Assertive validity axiom:

$$\Theta; \Rightarrow \top; \Upsilon$$

$\cap$ right:

$$\frac{\Theta; \Rightarrow A_1; \Upsilon \qquad \Theta; \Rightarrow A_2; \Upsilon}{\Theta; \Rightarrow A_1 \cap A_2; \Upsilon}$$

$\cap_i$ left: $(i = 0, 1)$

$$\frac{A_i, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}{A_0 \cap A_1, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

Hypotetical absurdity axiom:

$$\Theta; \bot \Rightarrow ; \Upsilon$$

Υ right:

$$\frac{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C_0, C_1}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, C_0 \curlyvee C_1}$$

Υ left:

$$\frac{\Theta_1; C_1 \Rightarrow ; \Upsilon_1 \qquad \Theta_2; C_2 \Rightarrow ; \Upsilon_2}{\Theta_1, \Theta_2; C_1 \curlyvee C_2 \Rightarrow ; \Upsilon_1, \Upsilon_2}$$

Sequent calculus for $\mathbf{BI}_p$

Implication and subtraction

$\supset$ right:

$$\frac{\Theta, A_1; \Rightarrow A_2; \Upsilon}{\Theta; \Rightarrow A_1 \supset A_2; \Upsilon}$$

$\supset$ left:

$$\frac{\Theta_1; \Rightarrow A_1; \Upsilon_1 \qquad A_2, \Theta_2; \epsilon \Rightarrow \epsilon'; \Upsilon_2}{A_1 \supset A_2, \Theta_1, \Theta_2; \epsilon \Rightarrow \epsilon'; \Upsilon_1, \Upsilon_2}$$

$\mathrel{-\!\!<}$ right:

$$\frac{\Theta_1; \epsilon \Rightarrow \epsilon'; \Upsilon_1, C_1 \qquad \Theta_2; C_2 \Rightarrow; \Upsilon_2}{\Theta_1, \Theta_2; \epsilon \Rightarrow \epsilon'; \Upsilon_1, \Upsilon_2, C_1 \mathrel{-\!\!<} C_2}$$

$\mathrel{-\!\!<}$ left:

$$\frac{\Theta; C_1 \Rightarrow; \Upsilon, C_2}{\Theta; C_1 \mathrel{-\!\!<} C_2 \Rightarrow; \Upsilon}$$

Categorical model for $\mathbf{BI}_p$

We show that categorical models of $\mathbf{BI}_p$ have the form of dialogue chirality.

We sketch the construction of the syntactic category:

- **objects** are formulas;
- **morphisms** are equivalence classes of sequent derivations;
- subject to naturality conditions [omitted].

Categorical model for $\mathbf{BI}_p$

- Let $\mathscr{A} = (\mathbf{Int}, \cap, \top)$ be the Cartesian category of intuitionistic formulas and derivations in $\mathbf{BI}_p$.
- Let $\mathscr{B} = (\mathbf{co-Int}, \curlyvee, \bot)$ be the monoidal category of co-intuitionistic formulas and derivations in $\mathbf{BI}_p$.
- We have contravariant operations

  $$\sim : \mathscr{A} \to \mathscr{A} \text{ (written } \sim_u) \text{ and } \frown : \mathscr{B} \to \mathscr{B} \text{ (written } {}_j\frown)$$

  Let $\Diamond(A) = {}_j \frown\frown A$ and $\boxdot(C) = \sim_u \sim C$.
- Define a functor $L = \Diamond : \mathscr{A} \to \mathscr{B}$ sending a derivation $d : A_1; \Rightarrow A_2;$ to the derivation $\Diamond d : ;\Diamond A_1 \Rightarrow ;\Diamond A_2$ defined in the obvious way.
  Similarly define a functor $R = \boxdot : \mathscr{B} \to \mathscr{A}$.
- $L \dashv R$: the unit and counit of the adjunction are given by the derivation of Proposition (ii).
- The duality $\frown$ is a contravariant monoidal functor $\mathscr{A} \to \mathscr{B}^{op}$, sending $d : A_1 \cap A_2; \Rightarrow A_3 \cap A_4;$ to $\frown d : ;\frown A_3 \curlyvee \frown A_4 \Rightarrow ;\frown A_1 \curlyvee A_2;$.

Categorical model for $\mathbf{BI}_p$

- Let $\langle A|C\rangle$ be the set of (equivalence classes of) sequent derivations of $A; \Rightarrow \boxdot C;$.
- $\mathscr{A}' = (\mathbf{Int}, \cap, \supset, \top)$ is in fact cartesian closed, so there is a natural bijection between $\mathscr{A}'(M \cap A, \boxdot C)$ and $\mathscr{A}'(A, M \supset \boxdot C)$.
- The provable equivalences of Proposition (iii) provide a natural bijection between $\mathscr{A}'(A, M \supset \boxdot C)$ and $\mathscr{A}'(A, \boxdot(\frown M \curlyvee C))$ (*"De Morgan definition" of* $\supset$).
- By composing, we obtain the family of natural bijections

$$\chi_{M,A,C} \colon \langle M \cap A|C\rangle \to \langle A| \frown M \curlyvee C\rangle.$$

Proposition

The following are provable in $\mathbf{BI}_p$*.*

(i) $\sim\frown A \iff A$ *and dually,* $C \iff \frown\sim C$.

(ii) $A\,;\ \Rightarrow\ \boxdot\,\Diamond A;$ *and* $;\,\Diamond\,\boxdot\,C\ \Rightarrow\ ;\ C$.

(iii) $M \supset \boxdot C \iff \boxdot((\frown M) \curlyvee C)$.

Proof for (ii) and (iii):

$$
\supset R\ \dfrac{\sim L\ \dfrac{\smallsetminus R\ \dfrac{;\Rightarrow \mathbf{u}\,;\mathbf{j} \qquad \frown L\ \dfrac{A\,;\Rightarrow A\,;}{A\,;\ \frown A \Rightarrow\ ;}}{A\,;\ \Rightarrow\ \mathbf{u}\ ;\ {}_j{\frown}{\frown} A}}{A, \sim_{j}{\frown}{\frown} A\ ;\ \Rightarrow\ \mathbf{u}\ ;}}{A\ ;\ \Rightarrow\ \underbrace{\sim_u\sim\ {}_j{\frown}{\frown} A}_{\boxdot\,\Diamond A};}
\qquad
\smallsetminus L\ \dfrac{\frown R\ \dfrac{\supset L\ \dfrac{\sim R\ \dfrac{;\,C \Rightarrow\,;\,C}{\Rightarrow\,\sim C\,;\,C} \qquad \mathbf{u}\,;\mathbf{j} \Rightarrow\ ;}{\sim_u\sim C\ ;\ \mathbf{j}\ \Rightarrow\ ;\ C}}{;\ \mathbf{j}\ \Rightarrow\ ;\ \frown\sim\sim C, C}}{\underbrace{{}_j{\frown}{\frown}\sim_u\sim C}_{\Diamond\,\boxdot\,C}\ ;\ \Rightarrow\ ;\ C}
$$

Categorical model for $\mathbf{BI}_p$

Proof for (ii) and (iii) (continues from the previous page):

$$
\supset\mathsf{L}\;\dfrac{M;\ \Rightarrow M\ ; \qquad \supset\mathsf{L}\;\dfrac{\sim\mathsf{R}\;\dfrac{;\ C\ \Rightarrow\ ;C}{;\ \Rightarrow\ \sim C\,;\,C} \qquad \mathbf{u}\ ;\ \Rightarrow \mathbf{u}\ ;}{\boxdot C\ ;\ \Rightarrow\ \mathbf{u}\ ;\ C}}{\curlyvee\mathsf{R}\;\dfrac{\frown\mathsf{R}\;\dfrac{M, M \supset \boxdot C\ ;\ \Rightarrow\ \mathbf{u}\ ;\ C}{M \supset \boxdot C\ ;\ \Rightarrow\ \mathbf{u}\ ;\ \frown M, C}}{\sim\mathsf{L}\;\dfrac{M \supset \boxdot C\ ;\ \Rightarrow\ \mathbf{u}\ ;\ (\frown M) \curlyvee C}{\supset\mathsf{R}\;\dfrac{M \supset \boxdot C, \sim((\frown M) \curlyvee C)\ ;\ \Rightarrow\ \mathbf{u}\ ;}{M \supset \boxdot C\ ;\ \Rightarrow\ \boxdot((\frown M) \curlyvee C)\ ;}}}}
$$

$$
\curlyvee\mathsf{L}\;\dfrac{\dfrac{M;\Rightarrow M;}{M\ ;\ \frown M\ \Rightarrow\ ;}\;\frown\mathsf{L} \qquad ;\ C\ \Rightarrow\ ;C}{\sim\mathsf{R}\;\dfrac{M\ ;\ (\frown M) \curlyvee C\ \Rightarrow\ ;\ C}{\supset\mathsf{L}\;\dfrac{M\ ;\ \Rightarrow \sim(\frown M \curlyvee C)\ ;\ C \qquad \mathbf{u}\ ;\ \Rightarrow\ \mathbf{u};}{\sim\mathsf{L}\;\dfrac{\sim_u \sim((\frown M) \curlyvee C), M\ ;\ \Rightarrow \mathbf{u}\ ;\ C}{\supset\mathsf{R}\;\dfrac{\boxdot((\frown M) \curlyvee C), M, \sim C\ ;\ \Rightarrow \mathbf{u};}{\supset\mathsf{R}\;\dfrac{\boxdot((\frown M) \curlyvee C), M\ ;\ \Rightarrow \boxdot C;}{\boxdot((\frown M) \curlyvee C)\ ;\ \Rightarrow M \supset \boxdot C;}}}}}}
$$

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